

RESOLUTIONS IN FACTORIZATION CATEGORIES

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ABSTRACT. Generalizing Eisenbud’s matrix factorizations, we define factorization categories. Following work of Positselski, we define their associated derived categories. We construct specific resolutions of factorizations built from a choice of resolutions of their components. We use these resolutions to lift fully-faithfulness statements from derived categories of Abelian categories to derived categories of factorizations and to construct a spectral sequence computing the morphism spaces in the derived categories of factorizations from Ext-groups of their components in the underlying Abelian category.

1. INTRODUCTION

Since their introduction by D. Eisenbud in [Eis80], matrix factorizations have spread from commutative algebra into a wide range of fields. In theoretical physics, M. Kontsevich realized that matrix factorizations represent boundary conditions in Landau-Ginzburg models. In algebraic geometry, deep statements nontrivially tying the geometry of projective hypersurfaces to matrix factorizations of their defining polynomial have been proven by D. Orlov [Orl09]. In addition, through mirror symmetry, matrix factorizations allow access to the structure of Fukaya categories of symplectic manifolds, [Sei08b, Ef09, AAEKO11, She11].

In recent years, generalizations of the homotopy category of matrix factorizations, [Buc86], have been considered. The most robust generalization is due to L. Positselski, [Pos09, Pos11], in terms of new derived categories associated to curved dg-modules. However, a certain useful piece of homological algebra was lacking for such categories. Namely, nice classes of resolutions.

We repackage Positselski’s ideas into the notion of a factorization category for a triple (\mathcal{A}, Φ, w) where \mathcal{A} is an Abelian category, $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ is an autoequivalence, $w : \text{Id} \rightarrow \Phi$ is a natural transformation. As factorization categories can rightly be viewed as a deformation of Φ -twisted, two-periodic chain complexes over \mathcal{A} , one should be able to build resolutions in a straightforward manner from resolutions of the components of a factorization. The main result of this paper is to provide a construction of such resolutions, see Theorems 3.1 and 3.4.

Now consider two triples as above, (\mathcal{A}, Φ, w) and (\mathcal{B}, Ψ, v) , and an additive functor, $\theta : \mathcal{A} \rightarrow \mathcal{B}$, such that

$$\theta \circ \Phi \cong \Psi \circ \theta$$

and

$$\theta(w_A) = v_{\theta(A)} : \theta(A) \rightarrow \theta(\Phi(A)) \cong \Psi(\theta(A)).$$

for all objects, $A \in \mathcal{A}$. Furthermore, assume that θ is left-exact, that \mathcal{A} has small coproducts and enough injectives, and that coproducts of injectives are injective. We can then use these resolutions to prove that if the right derived functor of θ is fully-faithful then so is a “right derived functor” associated to θ between the derived categories of factorizations.

Such statements prove useful in understanding derived categories of gauged Landau-Ginzburg models in algebraic geometry, [BFK12]. We can also use these resolutions to construct a spectral sequence computing the morphism spaces in the derived categories of factorizations whose E_1 -page consists of Ext-groups between the components of this factorization in the underlying Abelian category.

2. BASICS

Let \mathcal{A} be an Abelian category,

$$\Phi : \mathcal{A} \rightarrow \mathcal{A}$$

be an autoequivalence of \mathcal{A} , and

$$w : \text{Id}_{\mathcal{A}} \rightarrow \Phi$$

a natural transformation from the identity functor to Φ . We assume that

$$w_{\Phi(A)} = \Phi(w_A)$$

for all $A \in \mathcal{A}$.

Definition 2.1. A **factorization** of the triple, (A, Φ, w) , consists of a pair of objects of \mathcal{A} , E^{-1} and E^0 , and a pair of morphisms,

$$\begin{aligned} \phi_E^{-1} : \Phi^{-1}(E^0) &\rightarrow E^{-1} \\ \phi_E^0 : E^{-1} &\rightarrow E^0 \end{aligned}$$

such that

$$\begin{aligned} \phi_E^0 \circ \phi_E^{-1} &= \Phi^{-1}(w_{E^0}) : \Phi^{-1}(E^0) \rightarrow E^0, \\ \Phi(\phi_E^{-1}) \circ \phi_E^0 &= w_{E^{-1}} : E^{-1} \rightarrow \Phi(E^{-1}), \end{aligned}$$

We shall often simply denote the factorization, $(E^{-1}, E^0, \phi_E^{-1}, \phi_E^0)$, by E . The objects, E^0 and E^{-1} , are called the **components of the factorization**. We also set

$$E^i := \begin{cases} \Phi^j(E^0) & \text{if } i = 2j \\ \Phi^j(E^{-1}) & \text{if } i = 2j - 1. \end{cases}$$

A **morphism of factorizations**, $g : E \rightarrow F$, is a pair of morphisms in \mathcal{A} ,

$$\begin{aligned} g^{-1} : E^{-1} &\rightarrow F^{-1} \\ g^0 : E^0 &\rightarrow F^0, \end{aligned}$$

making the diagram,

$$\begin{array}{ccccc} \Phi^{-1}(E^0) & \xrightarrow{\phi_E^{-1}} & E^{-1} & \xrightarrow{\phi_E^0} & E^0 \\ \Phi^{-1}(g^0) \downarrow & & g^{-1} \downarrow & & g^0 \downarrow \\ \Phi^{-1}(F^0) & \xrightarrow{\phi_F^{-1}} & F^{-1} & \xrightarrow{\phi_F^0} & F^0 \end{array}$$

commute.

We let $\text{Fact}(w)$ be the category of factorizations.

Lemma 2.2. *The category, $\text{Fact}(w)$, is Abelian.*

Proof. For a morphism, $g : E \rightarrow F$, the componentwise kernel is naturally a factorization, as is the componentwise cokernel. This endows $\text{Fact}(w)$ with the structure of an Abelian category. \square

There is a natural notion of translation, or shift, of a factorization.

Definition 2.3. Let $[1]$ be the auto-equivalence of $\text{Fact}(w)$ defined as

$$\begin{aligned} [1] : \text{Fact}(w) &\rightarrow \text{Fact}(w) \\ E &\mapsto E[1] := (E^0, \Phi(E^{-1}), -\phi_E^0, -\Phi(\phi_E^{-1})) \\ g &\mapsto g[1] := (g^0, \Phi(g^{-1})). \end{aligned}$$

The functor, $[n]$, is the n -fold composition of $[1]$.

Definition 2.4. There is also a dg-category associated with factorizations. It is denoted by $\mathbf{Fact}(w)$. The objects are the same as $\text{Fact}(w)$. Given two factorizations, $E, F \in \mathbf{Fact}(w)$, we set

$$\text{Hom}_{\mathbf{Fact}(w)}^n(E, F) := \text{Hom}_w^n(E, F) := \text{Hom}_{\mathcal{A}}(E^{-1}, (F[n])^{-1}) \oplus \text{Hom}_{\mathcal{A}}(E^0, (F[n])^0).$$

The differential on $\text{Hom}_w^*(E, F)$ takes the pair, $g^{-1} : E^{-1} \rightarrow (F[n])^{-1}, g^0 : E^0 \rightarrow (F[n])^0$, to

$$\begin{aligned} &\phi_{F[n]}^0 \circ g^{-1} - (-1)^n g^0 \circ \phi_E^0 : E^{-1} \rightarrow F[n]^0 = F[n+1]^{-1} \\ &\Phi(\phi_{F[n]}^{-1}) \circ g^0 - (-1)^n \Phi(g^{-1}) \circ \Phi(\phi_E^{-1}) : E^0 \rightarrow \Phi(F[n]^{-1}) = F[n+1]^0. \end{aligned}$$

And a natural cone construction.

Definition 2.5. For any morphism, $g : E \rightarrow F$, we write, $C(g)$, for the factorization defined as

$$C(g) := \left(E^0 \oplus F^{-1}, \Phi(E^{-1}) \oplus F^0, \begin{pmatrix} -\phi_E^0 & 0 \\ g^{-1} & \phi_F^{-1} \end{pmatrix}, \begin{pmatrix} -\Phi(\phi_E^{-1}) & 0 \\ g^0 & \phi_F^0 \end{pmatrix} \right).$$

Definition 2.6. A **homotopy**, h , between two morphisms, $g_1, g_2 : E \rightarrow F$, is a pair of morphisms,

$$\begin{aligned} h^{-1} : E^{-1} &\rightarrow \Phi^{-1}(F^0) \\ h^0 : E^0 &\rightarrow F^{-1}, \end{aligned}$$

such that

$$\begin{aligned} g_1^{-1} - g_2^{-1} &= h^0 \circ \phi_E^0 + \phi_F^{-1} \circ h^{-1} \\ g_1^0 - g_2^0 &= \phi_F^0 \circ h^0 + \Phi(h^{-1}) \circ \Phi(\phi_E^{-1}). \end{aligned}$$

We let $K(\text{Fact}(w))$ be the homotopy category of $\text{Fact}(w)$. Note that the homotopy category of the dg-category, $\mathbf{Fact}(w)$, is $K(\text{Fact}(w))$.

Proposition 2.7. *The translation, $[1]$, and cones defined above give $K(\text{Fact}(w))$ the structure of a triangulated category.*

Proof. This is completely analogous to the standard proof that homotopy categories of chain complexes are triangulated so we refer the reader to Chapter 4 of [GM03]. \square

Definition 2.8. Let

$$E_s \xrightarrow{g_{s+1}} E_{s+1} \xrightarrow{g_{s+2}} \cdots \xrightarrow{g_0} E_0 \quad (2.1)$$

be a complex of factorizations, i.e. a sequence of morphisms in $\text{Fact}(w)$ satisfying,

$$g_{i+1} \circ g_i = 0$$

for all $s \leq i \leq -1$. We define a sequence of new factorizations inductively. We set

$$T_1 := C(g_0).$$

There is a natural morphism of factorizations,

$$\tilde{g}_i : E_i[-i-1] \rightarrow T_{i+1}.$$

We then set

$$T_i := C(\tilde{g}_i).$$

Finally, the **totalization** of the complex in Equation (2.1) is defined to be the factorization, T_{s+1} .

These definitions are due to Positselski, see loc.cit. and [Pos11].

Definition 2.9. A factorization is called **totally acyclic** if it lies in the smallest thick subcategory of $K(\text{Fact}(w))$ containing the totalizations of all exact complexes from $\text{Fact}(w)$. We let $\text{Acycl}(w)$ denote the thick subcategory of $K(\text{Fact}(w))$ consisting of acyclic factorizations. The **absolute derived category of factorizations** of the triple (\mathcal{A}, Φ, w) is the Verdier quotient,

$$D^{\text{abs}}(\text{Fact } w) := K(\text{Fact}(w)) / \text{Acycl}(w).$$

A morphism in $\text{Fact}(w)$ which becomes an isomorphism in $D^{\text{abs}}(\text{Fact } w)$ will be called a **quasi-isomorphism**, in analogy with the usual derived category. Similarly, two factorizations which are isomorphic in $D^{\text{abs}}(\text{Fact } w)$ are called **quasi-isomorphic**.

Definition 2.10. Assume that small coproducts exist in \mathcal{A} . A factorization is called **co-acyclic** if it lies in the smallest thick subcategory of $K(\text{Fact}(w))$ containing the totalizations of all exact complexes from $\text{Fact}(w)$ and closed under taking small coproducts. We let $\text{Co-acycl}(w)$ denote the thick subcategory of $K(\text{Fact}(w))$ consisting of acyclic factorizations. The **co-derived category of factorizations** of the triple (\mathcal{A}, Φ, w) is the Verdier quotient,

$$D^{\text{co}}(\text{Fact } w) := K(\text{Fact}(w)) / \text{Co-acycl}(w).$$

A morphism in $\text{Fact}(w)$ which becomes an isomorphism in $D^{\text{co}}(\text{Fact } w)$ will be called a **co-quasi-isomorphism**. Similarly, two factorizations which are isomorphic in $D^{\text{co}}(\text{Fact } w)$ are called **co-quasi-isomorphic**.

Definition 2.11. Assume that small products exist in \mathcal{A} . A factorization is called **contra-acyclic** if it lies in the smallest thick subcategory of $K(\text{Fact}(w))$ containing the totalizations of all exact complexes from $\text{Fact}(w)$ and closed under taking small products. We let $\text{Ctr-acycl}(w)$ denote the thick subcategory of $K(\text{Fact}(w))$ consisting of acyclic factorizations. The **contra-derived category of factorizations** of the triple (\mathcal{A}, Φ, w) is the Verdier quotient,

$$D^{\text{ctr}}(\text{Fact } w) := K(\text{Fact}(w)) / \text{Ctr-acycl}(w).$$

A morphism in $\text{Fact}(w)$ which becomes an isomorphism in $D^{\text{ctr}}(\text{Fact } w)$ will be called a **contra-quasi-isomorphism**. Similarly, two factorizations which are isomorphic in $D^{\text{co}}(\text{Fact } w)$ are called **contra-quasi-isomorphic**.

Remark 2.12. Let us attempt to provide some motivation for such definitions. Let us consider the derived category, $D(\mathcal{A})$. It is the localization of $K(\mathcal{A})$ at the class of quasi-isomorphisms. It can also be viewed as the Verdier quotient of $K(\mathcal{A})$ by acyclic complexes.

How does one make an acyclic complex? One way is to take an exact sequence of complexes,

$$0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0,$$

and totalize the complex to get an object of $\text{Ch}(\mathcal{A})$. This method of construction is fairly robust. Indeed, any finite acyclic complex is easily seen to be the totalization of an exact sequence of chain complexes. These are exactly the analogs of totally-acyclic factorizations. Thus, quotienting by totally-acyclic factorizations should be viewed as the analog of quotienting $K(\mathcal{A})$ by the thick subcategory of finite acyclic complexes.

To deal with unbounded complexes, we have to take some form of limit of totalizations of bounded exact complexes. Choice of direction of this limit naturally forces one to study infinite products or coproducts of bounded exact complexes. This connection motivates the definitions of co-acyclic and contra-acyclic complexes.

Lemma 2.13. *The categories, $D^{\text{abs}}(\text{Fact } w)$, $D^{\text{co}}(\text{Fact } w)$, $D^{\text{ctr}}(\text{Fact } w)$, with the the shift and triangles inherited from $K(\text{Fact}(w))$, are triangulated categories.*

Proof. Each of these categories is a Verdier quotient of a triangulated category by a thick triangulated subcategory hence triangulated [Ve77]. \square

Following the analogy with derived categories of Abelian categories, one can realize the various derived categories of factorizations as homotopy categories of factorizations with injective or projective components.

Lemma 2.14. *Let I be an object of $\text{Fact}(w)$ with I^{-1}, I^0 injective objects of \mathcal{A} . Let C be a co-acyclic factorization. Then,*

$$\text{Hom}_{K(\text{Fact}(w))}(C, I) = 0.$$

Let P be an object of $\text{Fact}(w)$ with P^{-1}, P^0 projective objects of \mathcal{A} . Let C be a contra-acyclic factorization. Then,

$$\text{Hom}_{K(\text{Fact}(w))}(P, C) = 0.$$

Proof. If $C_s, s \in S$ is a collection of objects left orthogonal to I , then $\bigoplus_{s \in S} C_s$ is also left orthogonal to I . We can reduce to checking that I is right orthogonal to totalizations of exact sequences. Any exact sequence is an iterated sequence of totalizations of short exact sequences. Thus, it suffices to check that I is left orthogonal to totalizations of short exact sequences.

Take a short exact sequence of factorizations,

$$0 \rightarrow E_1 \xrightarrow{g_1} E_2 \xrightarrow{g_2} E_3 \rightarrow 0.$$

Let C be the totalization of this short exact sequence. By definition, there is a triangle,

$$E_1[1] \xrightarrow{h} C(g_2) \rightarrow A \rightarrow E_1[2],$$

in $K(\text{Fact}(w))$. Therefore, there is a long exact sequence,

$$\begin{aligned} \cdots \rightarrow \text{Hom}_{K(\text{Fact}(w))}(C(g_2)[i+1], I) &\rightarrow \text{Hom}_{K(\text{Fact}(w))}(E_1[i+2], I) \rightarrow \text{Hom}_{K(\text{Fact}(w))}(C[i], I) \\ &\rightarrow \text{Hom}_{K(\text{Fact}(w))}(C(g_2)[i], I) \rightarrow \text{Hom}_{K(\text{Fact}(w))}(E_1[i+1], I) \rightarrow \cdots \end{aligned}$$

Showing that

$$\text{Hom}_{K(\text{Fact}(w))}(C[i], I) = 0$$

for all i is equivalent to showing that the maps,

$$\text{Hom}_{K(\text{Fact}(w))}(C(g_2)[i], I) \rightarrow \text{Hom}_{K(\text{Fact}(w))}(E_1[i+1], I),$$

are isomorphisms for all i .

There is a commutative diagram,

$$\begin{array}{ccccc} E_1 & \longrightarrow & E_2 & \xrightarrow{g_2} & E_3 \\ h[-1] \downarrow & & \downarrow & & \downarrow \\ C(g_2)[-1] & \longrightarrow & E_2 & \xrightarrow{g_2} & E_3 \end{array}$$

Apply $\text{Hom}_w^*(\bullet, I)$ to this diagram to get a commutative diagram of complexes,

$$\begin{array}{ccccc} \text{Hom}_w^*(E_3, I) & \longrightarrow & \text{Hom}_w^*(E_2, I) & \longrightarrow & \text{Hom}_w^*(C(g_2)[-1], I) \\ = \downarrow & & \downarrow & & h[-1] \downarrow \\ \text{Hom}_w^*(E_3, I) & \longrightarrow & \text{Hom}_w^*(E_2, I) & \longrightarrow & \text{Hom}_w^*(E_1, I) \end{array}$$

Since I has injective components, the sequence,

$$0 \rightarrow \text{Hom}_w^*(E_3, I) \rightarrow \text{Hom}_w^*(E_2, I) \rightarrow \text{Hom}_w^*(E_1, I) \rightarrow 0,$$

is an exact sequence of complexes.

Taking cohomology of all the complexes in the diagram above induces a morphism of long exact sequences,

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \text{Hom}_{K(\text{Fact}(w))}(E_3[i], I) & \longrightarrow & \text{Hom}_{K(\text{Fact}(w))}(E_2[i], I) & \longrightarrow & \text{Hom}_{K(\text{Fact}(w))}(C(g_2)[i-1], I) \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & h[i-1] \downarrow \\ \cdots & \longrightarrow & \text{Hom}_{K(\text{Fact}(w))}(E_3[i], I) & \longrightarrow & \text{Hom}_{K(\text{Fact}(w))}(E_2[i], I) & \longrightarrow & \text{Hom}_{K(\text{Fact}(w))}(E_1[i], I) \longrightarrow \cdots \end{array}$$

From the 5-lemma, we can conclude that $h[i]$ is an isomorphism for all i .

The proof for contra-acyclic and projective factorizations is completely analogous. \square

Let $\text{K}(\text{Inj } w)$ be the homotopy category of injective factorizations and let $\text{K}(\text{Proj } w)$ be the homotopy category of projective factorizations. The following two results use explicit resolutions constructed in the next section.

Proposition 2.15. *Assume that \mathcal{A} has small coproducts, enough injectives, and that coproducts of injectives are injective. The composition,*

$$\text{K}(\text{Inj } w) \rightarrow \text{K}(\text{Fact } w) \rightarrow \text{D}^{\text{co}}(\text{Fact } w),$$

is an exact equivalence of triangulated categories.

Assume that \mathcal{A} has small products, enough projectives and products of projectives are projective. The composition,

$$K(\text{Proj } w) \rightarrow K(\text{Fact } w) \rightarrow D^{\text{ctr}}(\text{Fact } w),$$

is an exact equivalence of triangulated categories.

Proof. Objects in $K(\text{Inj } w)$ lie in the right orthogonal to co-acyclic complexes by Lemma 2.14, which is, by definition, the kernel of the projection,

$$\pi : K(\text{Fact } w) \rightarrow D^{\text{co}}(\text{Fact } w).$$

A standard result, see Lemma 9.15 of [Nee01], states that π is fully-faithful on $K(\text{Inj } w)$. Theorem 3.1 states that any factorization is co-quasi-isomorphic to a factorization with injective components. Thus, π is essentially surjective.

The argument for projective factorizations runs the same with Theorem 3.4 in place of Theorem 3.1. \square

3. CONSTRUCTIONS OF THE RESOLUTIONS

In this section, we provide a useful method of replacing a factorization by a co-quasi-isomorphic factorization of injectives or by a contra-quasi-isomorphic factorization of projectives. We saw a few simple consequences of the existence of such replacements at the end of Section 2. In Section 4, we will present some more computationally-useful applications.

The construction of the resolutions starts with resolutions of the components. Let E be an object of $\text{Fact}(w)$. Choose injective resolutions of its components,

$$\begin{aligned} 0 &\longrightarrow E^{-1} \xrightarrow{d_0^{-1}} I_0^{-1} \xrightarrow{d_1^{-1}} I_1^{-1} \xrightarrow{d_2^{-1}} \dots \\ 0 &\longrightarrow E^0 \xrightarrow{d_0^0} I_0^0 \xrightarrow{d_1^0} I_1^0 \xrightarrow{d_2^0} \dots \end{aligned}$$

Assume that \mathcal{A} has small coproducts and define the following two objects of \mathcal{A} by combining even and odd components of the two resolutions:

$$\begin{aligned} I^{-1} &= \bigoplus_{2l} \Phi^{-l}(I_{2l}^{-1}) \oplus \bigoplus_{2l+1} \Phi^{-l-1}(I_{2l+1}^0) \\ I^0 &= \bigoplus_{2l} \Phi^{-l}(I_{2l}^0) \oplus \bigoplus_{2l+1} \Phi^{-l}(I_{2l+1}^{-1}). \end{aligned}$$

We wish to construct a factorization, $(I^{-1}, I^0, \phi_I^{-1}, \phi_I^0)$, using these objects as components. We will denote this, not-yet-defined, factorization as I . We will define the morphisms,

$$\begin{aligned} \phi_I^{-1} &: \Phi^{-1}(I^0) \rightarrow I^{-1} \\ \phi_I^0 &: I^{-1} \rightarrow I^0, \end{aligned}$$

in terms of their components,

$$\begin{aligned}
\phi_{2l+1,2j+1}^{-1} &: \Phi^{-l-1}(I_{2l+1}^{-1}) \rightarrow \Phi^{-1}(I^0) \xrightarrow{\phi_I^{-1}} I^{-1} \rightarrow \Phi^{-j-1}(I_{2j+1}^0) \\
\phi_{2l+1,2j}^{-1} &: \Phi^{-l-1}(I_{2l+1}^{-1}) \rightarrow \Phi^{-1}(I^0) \xrightarrow{\phi_I^{-1}} I^{-1} \rightarrow \Phi^{-j}(I_{2j}^{-1}) \\
\phi_{2l,2j+1}^{-1} &: \Phi^{-l-1}(I_{2l}^0) \rightarrow \Phi^{-1}(I^0) \xrightarrow{\phi_I^{-1}} I^{-1} \rightarrow \Phi^{-j-1}(I_{2j+1}^0) \\
\phi_{2l,2j}^{-1} &: \Phi^{-l-1}(I_{2l}^0) \rightarrow \Phi^{-1}(I^0) \xrightarrow{\phi_I^{-1}} I^{-1} \rightarrow \Phi^{-j}(I_{2j}^{-1})
\end{aligned}$$

and

$$\begin{aligned}
\phi_{2l+1,2j+1}^0 &: \Phi^{-l-1}(I_{2l+1}^0) \rightarrow I^{-1} \xrightarrow{\phi_I^0} I^0 \rightarrow \Phi^{-j}(I_{2j+1}^{-1}) \\
\phi_{2l+1,2j}^0 &: \Phi^{-l-1}(I_{2l+1}^0) \rightarrow I^{-1} \xrightarrow{\phi_I^0} I^0 \rightarrow \Phi^{-j}(I_{2j}^0) \\
\phi_{2l,2j+1}^0 &: \Phi^{-l}(I_{2l}^{-1}) \rightarrow I^{-1} \xrightarrow{\phi_I^0} I^0 \rightarrow \Phi^{-j}(I_{2j+1}^{-1}) \\
\phi_{2l,2j}^0 &: \Phi^{-l}(I_{2l}^{-1}) \rightarrow I^{-1} \xrightarrow{\phi_I^0} I^0 \rightarrow \Phi^{-j}(I_{2j}^0).
\end{aligned}$$

Requiring that

$$\begin{aligned}
\phi_I^0 \circ \phi_I^{-1} &= w_{\Phi^{-1}(I^0)} \\
\Phi(\phi_I^{-1}) \circ \phi_I^0 &= w_{I^{-1}}
\end{aligned}$$

is equivalent to requiring the following identities of the components of ϕ_I^{-1} and ϕ_I^0 :

$$\sum_{t \in \mathbb{Z}} \phi_{t,q}^0 \circ \phi_{p,t}^{-1} = \begin{cases} 0 & p \neq q \\ w & p = q \end{cases}$$

and

$$\sum_{t \in \mathbb{Z}} \Phi(\phi_{t,q}^{-1}) \circ \phi_{p,t}^0 = \begin{cases} 0 & p \neq q \\ w & p = q. \end{cases}$$

In our construction, only finitely many terms in these sums will be nonzero leaving their definition completely unambiguous.

The construction of the components will involve choosing lifts of ϕ_E^{-1} and ϕ_E^0 to the specified injective resolutions. Such choices, of course, always exist. However, for certain applications, we will need to work with specific choices of such lifts. As such, it is useful to specify choices of lifts,

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Phi^{-1}(E^0) & \xrightarrow{\Phi^{-1}(d_0^0)} & \Phi^{-1}(I_0^0) & \xrightarrow{\Phi^{-1}(d_1^0)} & \Phi^{-1}(I_1^0) \xrightarrow{\Phi^{-1}(d_2^0)} \dots \\
& & \phi_E^{-1} \downarrow & & \phi_0^{-1} \downarrow & & \phi_1^{-1} \downarrow \\
0 & \longrightarrow & E^{-1} & \xrightarrow{d_0^{-1}} & I_0^{-1} & \xrightarrow{d_1^{-1}} & I_1^{-1} \xrightarrow{d_2^{-1}} \dots \\
& & \phi_E^0 \downarrow & & \phi_0^0 \downarrow & & \phi_1^0 \downarrow \\
0 & \longrightarrow & E^0 & \xrightarrow{d_0^0} & I_0^0 & \xrightarrow{d_1^0} & I_1^0 \xrightarrow{d_2^0} \dots
\end{array}$$

Since $\phi_E^0 \circ \phi_E^{-1} = w$ and $\Phi(\phi_E^{-1}) \circ \phi_E^0 = w$, the compositions of the lifts to the injective resolutions are homotopic to w . It will also be useful to specify the homotopies beforehand.

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \Phi^{-1}(E^0) & \xrightarrow{\Phi^{-1}(d_0^0)} & \Phi^{-1}(I_0^0) & \xrightarrow{\Phi^{-1}(d_1^0)} & \Phi^{-1}(I_1^0) & \xrightarrow{\Phi^{-1}(d_2^0)} & \Phi^{-1}(I_2^0) & \xrightarrow{\Phi^{-1}(d_3^0)} & \dots \\
& & \downarrow 0 & & \downarrow \beta_0^0 & \nearrow h_0^0 & \downarrow \beta_1^0 & \nearrow h_1^0 & \downarrow \beta_2^0 & \nearrow h_2^0 & \\
0 & \longrightarrow & E^0 & \xrightarrow{d_0^0} & I_0^0 & \xrightarrow{d_1^0} & I_1^0 & \xrightarrow{d_1^0} & I_2^0 & \xrightarrow{d_3^0} & \dots
\end{array}$$

where $\beta_i^0 = w_{\Phi^{-1}(I_j^0)} - \phi_j^0 \circ \phi_j^{-1}$, and

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & E^{-1} & \xrightarrow{d_0^{-1}} & I_0^{-1} & \xrightarrow{d_1^{-1}} & I_1^{-1} & \xrightarrow{d_2^{-1}} & I_2^{-1} & \xrightarrow{d_3^{-1}} & \dots \\
& & \downarrow 0 & & \downarrow \beta_0^{-1} & \nearrow h_0^{-1} & \downarrow \beta_1^{-1} & \nearrow h_1^{-1} & \downarrow \beta_2^{-1} & \nearrow h_2^{-1} & \\
0 & \longrightarrow & \Phi(E^{-1}) & \xrightarrow{\Phi(d_0^{-1})} & \Phi(I_0^{-1}) & \xrightarrow{\Phi(d_1^{-1})} & \Phi(I_1^{-1}) & \xrightarrow{\Phi(d_2^{-1})} & \Phi(I_2^{-1}) & \xrightarrow{\Phi(d_3^{-1})} & \dots
\end{array}$$

where $\beta_i^{-1} = w_{I_j^{-1}} - \Phi(\phi_j^{-1}) \circ \phi_j^0$.

Finally, we state the main result of the paper.

Theorem 3.1. *Assume that small coproducts exist in \mathcal{A} and that \mathcal{A} has enough injective objects.*

Let E be an object of $\text{Fact}(w)$. Choose injective resolutions of its components, lifts of ϕ_E^{-1} and ϕ_E^0 to these injective resolutions, and null-homotopies of the difference of w and the compositions of the lifts as above. Define I^{-1} and I^0 as above.

There exists a factorization, $I = (I^{-1}, I^0, \phi_I^{-1}, \phi_I^0)$, and a co-quasi-isomorphism, $d_0 : E \rightarrow I$, such that

- *We have $\phi_{p,q}^{-1} = \phi_{p,q}^0 = 0$ for $q > p + 1$.*
- *We have equalities,*

$$\begin{aligned}
\Phi^{-l-1}(d_{2l+2}^{-1}) &= \phi_{2l+1,2l+2}^{-1} : \Phi^{-l-1}(I_{2l+1}^{-1}) \rightarrow \Phi^{-l-1}(I_{2l+2}^{-1}) \\
-\Phi^{-l-1}(d_{2l+1}^0) &= \phi_{2l,2l+1}^{-1} : \Phi^{-l-1}(I_{2l}^0) \rightarrow \Phi^{-l-1}(I_{2l+1}^0) \\
-\Phi^{-l-1}(d_{2l+2}^0) &= \phi_{2l+1,2l+2}^0 : \Phi^{-l-1}(I_{2l+1}^0) \rightarrow \Phi^{-l-1}(I_{2l+2}^0) \\
\Phi^{-l}(d_{2l+1}^{-1}) &= \phi_{2l,2l+1}^0 : \Phi^{-l}(I_{2l}^{-1}) \rightarrow \Phi^{-l}(I_{2l+1}^{-1}),
\end{aligned}$$

and

$$\begin{aligned}
\Phi^{-l-1}(\phi_{2l+1}^0) &= \phi_{2l+1,2l+1}^{-1} : \Phi^{-l-1}(I_{2l+1}^{-1}) \rightarrow \Phi^{-l-1}(I_{2l+1}^0) \\
\Phi^{-l}(\phi_{2l}^{-1}) &= \phi_{2l,2l}^{-1} : \Phi^{-l-1}(I_{2l}^0) \rightarrow \Phi^{-l}(I_{2l}^{-1}) \\
\Phi^{-l}(\phi_{2l+1}^{-1}) &= \phi_{2l+1,2l+1}^0 : \Phi^{-l-1}(I_{2l+1}^0) \rightarrow \Phi^{-l}(I_{2l+1}^{-1}) \\
\Phi^{-l}(\phi_{2l}^0) &= \phi_{2l,2l}^0 : \Phi^{-l}(I_{2l}^{-1}) \rightarrow \Phi^{-l}(I_{2l}^0),
\end{aligned}$$

and

$$\begin{aligned}
\Phi^{-l-1}(h_{2l}^{-1}) &= \phi_{2l+1,2l}^{-1} : \Phi^{-l-1}(I_{2l+1}^{-1}) \rightarrow \Phi^{-l}(I_{2l}^{-1}) \\
-\Phi^{-l}(h_{2l-1}^0) &= \phi_{2l,2l-1}^{-1} : \Phi^{-l-1}(I_{2l}^0) \rightarrow \Phi^{-l}(I_{2l-1}^0) \\
-\Phi^{-l}(h_{2l}^0) &= \phi_{2l+1,2l}^0 : \Phi^{-l-1}(I_{2l+1}^0) \rightarrow \Phi^{-l}(I_{2l}^0) \\
\Phi^{-l}(h_{2l-1}^{-1}) &= \phi_{2l,2l-1}^0 : \Phi^{-l}(I_{2l}^{-1}) \rightarrow \Phi^{-l+1}(I_{2l-1}^{-1}).
\end{aligned}$$

- d_0 is given by the compositions,

$$\begin{aligned}
E^{-1} &\xrightarrow{d_0^{-1}} I_0^{-1} \rightarrow I^{-1} \\
E^0 &\xrightarrow{d_0^0} I_0^0 \rightarrow I^0.
\end{aligned}$$

- d_0 is a quasi-isomorphism when both injective resolutions are finite.

Proof. We will construct $\phi_{p,q}^{-1}$ and $\phi_{p,q}^0$ such that

$$\sum_{t \in \mathbb{Z}} \phi_{t,p+n}^0 \circ \phi_{p,t}^{-1} = \begin{cases} 0 & n \neq 0 \\ w & n = 0 \end{cases} \quad (3.1)$$

and

$$\sum_{t \in \mathbb{Z}} \Phi(\phi_{t,p+n}^{-1}) \circ \phi_{p,t}^0 = \begin{cases} 0 & n \neq 0 \\ w & n = 0. \end{cases} \quad (3.2)$$

We will proceed by downward induction on n . We begin by defining $\phi_{p,q}^{-1}$ and $\phi_{p,q}^0$ for $p-1 \leq q$ exactly as in the conclusions of the theorem. This satisfies the cases, $n = 2, 1, 0$, of Equations (3.1) and (3.2).

Now assume we have constructed $\phi_{p,q}^{-1}$ and $\phi_{p,q}^0$ for $q \geq p-m$ satisfying Equations (3.1) and (3.2) for $n \geq -m+1$. We need to construct $\phi_{s,s-m-1}^{-1}$ and $\phi_{s,s-m-1}^0$ such that

$$\sum_{p-m \leq t \leq p} \phi_{t,p-m}^0 \circ \phi_{p,t}^{-1} + \phi_{p-m-1,p-m}^0 \circ \phi_{p,p-m-1}^{-1} + \phi_{p+1,p-m}^0 \circ \phi_{p,p+1}^{-1} = 0 \quad (3.3)$$

and

$$\sum_{p-m \leq t \leq p} \Phi(\phi_{t,p-m}^{-1}) \circ \phi_{p,t}^0 + \Phi(\phi_{p-m-1,p-m}^{-1}) \circ \phi_{p,p-m-1}^0 + \Phi(\phi_{p+1,p-m}^{-1}) \circ \phi_{p,p+1}^0 = 0. \quad (3.4)$$

We will see that solving Equation (3.3) and (3.4) amounts to choosing a null-homotopy for an acyclic chain map between complexes of injectives. Solving Equation (3.3) for p even and Equation (3.2) for p odd is independent from solving Equation (3.3) for p odd and Equation (3.2) for p even. We will solve Equation (3.3) for p even and Equation (3.2) for p odd. The other case is completely analogous.

Assume that $m = 2r$. The case of odd m follows analogously. Consider the chain complexes of injectives, $(\Phi^{-u-1}(I_v^0), \Phi^{-u-1}(d_v^0))$ and $(\Phi^{-u-r}(I_{v-m}^0), \Phi^{-u-r}(d_{v-m}^0))$. Each complex contains homology in a single degree, 0 for $(\Phi^{-u-1}(I_v^0), \Phi^{-u-1}(d_v^0))$ and m for $(\Phi^{-u-r}(I_{v-m}^0), \Phi^{-u-r}(d_{v-m}^0))$.

There are morphisms,

$$\begin{aligned}\psi_{u,2q} &:= \Phi^{q-u} \left(\sum_{2q-m < t \leq 2q} \phi_{t,2q-m+1}^0 \circ \phi_{2q,t}^{-1} \right) : \Phi^{-u-1}(I_{2q}^0) \rightarrow \Phi^{-u-r}(I_{2q-m}^0) \\ \psi_{u,2q+1} &:= \Phi^{q-u} \left(\sum_{2q+1-m < t \leq 2q+1} \Phi(\phi_{t,2q-m+1}^{-1}) \circ \phi_{2q+1,t}^0 \right) : \Phi^{-u-1}(I_{2q+1}^0) \rightarrow \Phi^{-u-r}(I_{2q-m+1}^0).\end{aligned}$$

We claim that $\psi_u : (\Phi^{-u-1}(I_v^0), \Phi^{-u-1}(d_v^0)) \rightarrow (\Phi^{-u-r}(I_{v-m}^0), \Phi^{-u-r}(d_{v-m}^0))$ is a chain map. Let us assume the validity of this claim for the moment and continue. Since ψ_u must induce the trivial map on the homology of the complexes and the components of the complexes are injectives, there exists a null-homotopy,

$$h_{u,v} : \Phi^{-u-1}(I_v^0) \rightarrow \Phi^{-u-r}(I_{v-m-1}^0),$$

of ψ_u . Let us draw the diagram for the homotopy. Recall that $\phi_{2q,2q+1}^{-1} = -\Phi^{-q-1}(d_{2q+1}^0)$ and $\phi_{2q-1,2q}^0 = -\Phi^{-q}(d_{2q}^0)$.

$$\begin{array}{ccccccc} \Phi^{-u-1}(I_{2q-1}^0) & \longrightarrow & \Phi^{-u-1}(I_{2q}^0) & \xrightarrow{-\Phi^{q-u}(\phi_{2q,2q+1}^{-1})} & \Phi^{-u-1}(I_{2q+1}^0) & \xrightarrow{-\Phi^{q-u}(\phi_{2q+1,2q+2}^0)} & \Phi^{-u-1}(I_{2q+2}^0) \\ & \searrow h_{u,2q} & \downarrow \psi_{u,2q} & & \downarrow \psi_{u,2q+1} & \searrow h_{u,2q+2} & \\ \Phi^{-u-r}(I_{2q-m-1}^0) & \xrightarrow{-\Phi^{q-u}(\phi_{2q-m-1,2q-m}^0)} & \Phi^{-u-r}(I_{2q-m}^0) & \xrightarrow{-\Phi^{q-u}(\phi_{2q-m,2q-m+1}^{-1})} & \Phi^{-u-r}(I_{2q-m+1}^0) & \longrightarrow & \Phi^{-u-r}(I_{2q-m+2}^0) \end{array}$$

We can rewrite the equations for the homotopy,

$$\begin{aligned}\psi_{u,2q} &= -h_{u,2q+1} \circ \Phi^{q-u}(\phi_{2q,2q+1}^{-1}) - \Phi^{q-u}(\phi_{2q-m-1,2q-m}^0) \circ h_{u,2q} \\ \psi_{u,2q+1} &= -h_{u,2q+2} \circ \Phi^{q-u}(\phi_{2q+1,2q+2}^0) - \Phi^{q-u+1}(\phi_{2q-m,2q-m+1}^{-1}) \circ h_{u,2q+1},\end{aligned}$$

as

$$\begin{aligned}\sum_{2q-m < t \leq 2q} \phi_{t,2q-m+1}^0 \circ \phi_{2q,t}^{-1} + \Phi^{u-q}(h_{u,2q+1}) \circ \phi_{2q,2q+1}^{-1} + \phi_{2q-m-1,2q-m}^0 \circ \Phi^{u-q}(h_{u,2q}) &= 0 \\ \sum_{2q-m+1 < t \leq 2q+1} \Phi(\phi_{t,2q-m+1}^{-1}) \circ \phi_{2q+1,t}^0 + \Phi(\phi_{2q-m,2q-m+1}^{-1}) \circ \Phi^{u-q}(h_{u,2q+1}) \\ + \Phi^{u-q+1}(h_{u,2q+2}) \circ \phi_{2q+1,2q+2}^0 &= 0.\end{aligned}$$

We then set

$$\begin{aligned}\phi_{2q+1,2q-m}^0 &:= \Phi^{u-q}(h_{u,2q+1}) \\ \phi_{2q,2q-m-1}^{-1} &:= \Phi^{u-q}(h_{u,2q})\end{aligned}$$

to solve Equation (3.3) for $p = 2q$ and Equation (3.4) for $p = 2q + 1$.

Thus, we have constructed $\phi_{s,s-m-1}^{-1}$ and $\phi_{s,s-m-1}^0$ completing the induction step if we can show that ψ_u is a chain map. We check the commutativity of the square,

$$\begin{array}{ccc}
\Phi^{-u-1}(I_{2q}^0) & \xrightarrow{-\Phi^{q-u}(\phi_{2q,2q+1}^{-1})} & \Phi^{-u-1}(I_{2q+1}^0) \\
\downarrow \psi_{u,2q} & & \downarrow \psi_{u,2q+1} \\
\Phi^{-u-r}(I_{2q-m+1}^0) & \xrightarrow{-\Phi^{q-u}(\phi_{2q-m,2q-m+1}^{-1})} & \Phi^{-u-r}(I_{2q+m+1}^0),
\end{array}$$

as the other squares are handled similarly. Commutativity of the above square is equivalent to the equality,

$$\left(\sum_{2q-m+1 \leq t \leq 2q+1} \Phi(\phi_{t,2q-m+1}^{-1}) \circ \phi_{2q+1,t}^0 \right) \circ \phi_{2q,2q+1}^{-1} = \Phi(\phi_{2q-m,2q+m-1}^{-1}) \circ \left(\sum_{2q-m \leq t \leq 2q} \phi_{t,2q-m}^0 \circ \phi_{2q,t}^{-1} \right). \quad (3.5)$$

From the induction hypothesis, for $2q-m+1 \leq t \leq 2q+1$, we have

$$\phi_{2q+1,t}^0 \circ \phi_{2q,2q+1}^{-1} = - \sum_{t-1 \leq s \leq 2q} \phi_{s,t}^0 \circ \phi_{2q,s}^{-1},$$

and, for $2q-m \leq t \leq 2q$,

$$\Phi(\phi_{2q-m,2q+m-1}^{-1}) \circ \phi_{t,2q-m}^0 = - \sum_{2q-m-1 \leq s \leq t+1} \Phi(\phi_{s,2q+m-1}^{-1}) \circ \phi_{t,s}^0.$$

Thus, both sides of Equation (3.5) are equal to

$$- \sum_{2q-m-1 \leq s \leq t+1 \leq 2q} \Phi(\phi_{s,2q+m-1}^{-1}) \circ \phi_{t,s}^0 \circ \phi_{2q,t}^{-1}.$$

This finishes the construction of the factorization, $(I^{-1}, I^0, \phi_I^{-1}, \phi_I^0)$.

We turn to checking that d_0 , as defined in the conclusion of the theorem, is a morphism of factorizations. By the construction of ϕ_I^{-1} and ϕ_I^0 , to check that the bolded squares in

$$\begin{array}{ccccc}
\Phi^{-1}(E^0) & \xrightarrow{\phi_E^{-1}} & E^{-1} & \xrightarrow{\phi_E^0} & E^0 \\
\downarrow \Phi^{-1}(d_0^0) & & \downarrow (d_0^{-1} \ 0) & & \downarrow (d_0^0 \ 0 \ 0) \\
\Phi^{-1}(I_0^0) & \xrightarrow{(\phi_0^{-1} \ \Phi^{-1}(d_0^{-1}))} & I_0^{-1} \oplus \Phi^{-1}(I_1^0) & \xrightarrow{\begin{pmatrix} \phi_0^0 & h_0^0 \\ d_1^{-1} & \phi_1^{-1}(d_2^0) \end{pmatrix}} & I_0^0 \oplus I_1^{-1} \oplus \Phi^{-1}(I_2^0) \\
\downarrow & & \downarrow & & \downarrow \\
\Phi^{-1}(I_0^0) & \xrightarrow{\phi_I^{-1}} & I_0^{-1} & \xrightarrow{\phi_I^0} & I_0^0
\end{array}$$

commute, it suffices to show that the upper squares commute. This is immediate.

Finally, we demonstrate that the cone of d_0 is co-acyclic. For those used to derived categories, the idea is quite simple; the cone of the morphism behaves like the sum of the two good truncations of the injective resolutions, hence, like a complex with no cohomology. Morally, this complex is then split into short exact sequences. In the language of factorizations this amounts to constructing the complex as a colimit of totalizations, which is finite

when the resolutions are finite. Let H be the cokernel of d_0 . The factorization, H , and the cone of d_0 differ by a totally acyclic complex as d_0 is a monomorphism. It suffices to check that H is co-acyclic/totally acyclic.

Let C_j^0 be the cokernel of $d_j^0 : I_{j-1}^0 \rightarrow I_j^0$ and let C_j^{-1} be the cokernel of $d_j^{-1} : I_{j-1}^{-1} \rightarrow I_j^{-1}$. From exactness, C_j^0 is the kernel of d_{j+1}^0 and C_j^{-1} is the kernel of d_{j+1}^{-1} and we have exact sequences,

$$\begin{aligned} 0 \rightarrow C_{j-1}^{-1} \rightarrow I_{j-1}^{-1} \xrightarrow{d_j^{-1}} I_j^{-1} \rightarrow C_j^{-1} \rightarrow 0 \\ 0 \rightarrow C_{j-1}^0 \rightarrow I_{j-1}^0 \xrightarrow{d_j^0} I_j^0 \rightarrow C_j^0 \rightarrow 0. \end{aligned}$$

Consider the subfactorization, $\tau_{\leq s} H$, of H given by restricting the components to their good truncations. The factorization, $\tau_{\leq s} H$, has components,

$$\begin{aligned} \tau_{\leq j} H^0 &= C_0^0 \oplus \bigoplus_{0 < 2l < j} \Phi^{-l}(I_{2l}^0) \oplus \bigoplus_{0 < 2l+1 < j} \Phi^{-l}(I_{2l+1}^{-1}) \oplus \begin{cases} \Phi^{-t}(C_j^0) & j = 2t \\ \Phi^{-t}(C_j^{-1}) & j = 2t + 1, \end{cases} \\ \tau_{\leq j} H^{-1} &= C_0^{-1} \oplus \bigoplus_{0 < 2l+1 < j} \Phi^{-l-1}(I_{2l+1}^0) \oplus \bigoplus_{0 < 2l < j} \Phi^{-l}(I_{2l}^{-1}) \oplus \begin{cases} \Phi^{-t}(C_j^{-1}) & j = 2t \\ \Phi^{-t-1}(C_j^0) & j = 2t + 1, \end{cases} \end{aligned}$$

and morphisms between components induced by H . Let S_i denote the factorization with components,

$$\begin{aligned} S_j^0 &= \begin{cases} \Phi^{-t+1}(C_j^{-1}) \oplus \Phi^{-t}(C_j^0) & j = 2t \\ \Phi^{-t}(C_j^0) \oplus \Phi^{-t}(C_j^{-1}) & j = 2t + 1 \end{cases} \\ S_j^{-1} &= \begin{cases} \Phi^{-t}(C_j^0) \oplus \Phi^{-t}(C_j^{-1}) & j = 2t \\ \Phi^{-t}(C_j^{-1}) \oplus \Phi^{-t-1}(C_j^0) & j = 2t + 1 \end{cases} \end{aligned}$$

and morphisms

$$\begin{aligned} \phi_{S_j}^0 &= \begin{cases} \begin{pmatrix} 0 & w_{\Phi^{-t}(C_j^{-1})} \\ \text{id}_{\Phi^{-t}(C_j^0)} & 0 \end{pmatrix} & j = 2l \\ \begin{pmatrix} 0 & w_{\Phi^{-t-1}(C_j^0)} \\ \text{id}_{\Phi^{-t}(C_j^{-1})} & 0 \end{pmatrix} & j = 2l + 1 \end{cases} \\ \phi_{S_j}^{-1} &= \begin{cases} \begin{pmatrix} 0 & w_{\Phi^{-t-1}(C_j^0)} \\ \text{id}_{\Phi^{-t}(C_j^{-1})} & 0 \end{pmatrix} & j = 2l \\ \begin{pmatrix} 0 & w_{\Phi^{-t-1}(C_j^{-1})} \\ \text{id}_{\Phi^{-t-1}(C_j^0)} & 0 \end{pmatrix} & j = 2l + 1 \end{cases} \end{aligned}$$

Note that S_j is manifestly a null-homotopic factorization. There are a short exact sequences,

$$0 \rightarrow \tau_{\leq j} \mathcal{H} \rightarrow \tau_{\leq j+1} \mathcal{H} \rightarrow S_{j+1} \rightarrow 0,$$

of factorizations.

Thus, in $\text{D}^{\text{abs}}(\text{Fact } w)$, $\tau_{\leq j} H$ and $\tau_{\leq j+1} H$ are isomorphic for $j \geq 0$. If the resolutions are finite, we see that, since $\tau_{\leq j} H = 0$ for $j \gg 0$, H is totally-acyclic.

In general, the colimit of these morphisms is isomorphic to H . As we can write the colimit via the short exact sequence,

$$0 \rightarrow \bigoplus_{j \geq 0} \tau_{\leq j} H \rightarrow \bigoplus_{j \geq 0} \tau_{\leq j} H \rightarrow \operatorname{colim} \tau_{\leq j} H = H \rightarrow 0,$$

we see that H is co-acyclic in general. \square

There is a special situation where the components, $\phi_{p,q}^{-1}$ and $\phi_{p,q}^0$, vanish for $q < p - 1$.

Corollary 3.2. *Assume that*

$$\begin{aligned} h_p^{-1} \circ \phi_{p+1}^{-1} &= \Phi(\phi_p^{-1}) \circ h_p^0 \\ \Phi(h_p^0) \circ \phi_{p+1}^0 &= \Phi(\phi_p^0) \circ h_p^{-1} \\ \Phi(h_{p-1}^{-1}) \circ h_p^{-1} &= 0 \\ \Phi(h_{p-1}^0) \circ h_p^0 &= 0. \end{aligned}$$

Then, in the factorization constructed in Theorem 3.1, we may take

$$\phi_{p,q}^{-1} = \phi_{p,q}^0 = 0$$

for $q < p - 1$.

Proof. Under the hypotheses, we can take $\phi_{p,q}^{-1} = \phi_{p,q}^0 = 0$ for $q < p - 1$ and satisfy Equations (3.1) and (3.2) for all n . \square

Remark 3.3. The situation where Corollary 3.2 is most applicable is where one of E^{-1} or E^0 is 0. For specificity, let us say that $E^{-1} = 0$. Then the hypotheses of Corollary 3.2 are equivalent to the null-homotopy, h_{\bullet}^0 , of the action of w on I_{\bullet}^0 squaring to zero. This occurs, in the dual situation, for Koszul factorizations, see [Eis80].

We also have the dual statement which we record in full detail for ease of future reference. Assume that \mathcal{A} has enough projectives. Let E be an object of $\operatorname{Fact}(w)$. Choose projective resolutions of its components and lifts of ϕ_E^{-1} and ϕ_E^0 to the those resolutions,

$$\begin{array}{ccccccc} \dots & \xrightarrow{\Phi^{-1}(d_{-2}^0)} & \Phi^{-1}(P_{-1}^0) & \xrightarrow{\Phi^{-1}(d_{-1}^0)} & \Phi^{-1}(P_0^0) & \xrightarrow{\Phi^{-1}(d_0^0)} & \Phi^{-1}(E^0) \longrightarrow 0 \\ & & \downarrow \phi_{-1}^{-1} & & \downarrow \phi_0^{-1} & & \downarrow \phi_E^{-1} \\ \dots & \xrightarrow{d_{-2}^{-1}} & P_{-1}^{-1} & \xrightarrow{d_{-1}^{-1}} & P_0^{-1} & \xrightarrow{d_0^{-1}} & E^{-1} \longrightarrow 0 \\ & & \downarrow \phi_{-1}^0 & & \downarrow \phi_0^0 & & \downarrow \phi_E^0 \\ \dots & \xrightarrow{d_{-2}^0} & P_{-1}^0 & \xrightarrow{d_{-1}^0} & P_0^0 & \xrightarrow{d_0^0} & E^0 \longrightarrow 0. \end{array}$$

Also choose null-homotopies,

$$\begin{array}{ccccccc} \dots & \xrightarrow{\Phi^{-1}(d_{-3}^0)} & \Phi^{-1}(P_{-2}^0) & \xrightarrow{\Phi^{-1}(d_{-2}^0)} & \Phi^{-1}(P_{-1}^0) & \xrightarrow{\Phi^{-1}(d_{-1}^0)} & \Phi^{-1}(P_0^0) & \xrightarrow{\Phi^{-1}(d_0^0)} & \Phi^{-1}(E^0) \longrightarrow 0 \\ & \swarrow h_{-3}^0 & \downarrow \beta_{-2}^0 & \swarrow h_{-2}^0 & \downarrow \beta_{-1}^0 & \swarrow h_{-1}^0 & \downarrow \beta_0^0 & \downarrow 0 & \\ \dots & \xrightarrow{d_{-3}^0} & P_{-2}^0 & \xrightarrow{d_{-2}^0} & P_{-1}^0 & \xrightarrow{d_{-1}^0} & P_0^0 & \xrightarrow{d_0^0} & E^0 \longrightarrow 0 \end{array}$$

where $\beta_j^0 = w_{\Phi^{-1}(P_j^0)} - \phi_j^0 \circ \phi_j^{-1}$, and

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{d_{-3}^{-1}} & P_{-2}^{-1} & \xrightarrow{d_{-2}^{-1}} & P_{-1}^{-1} & \xrightarrow{d_{-1}^{-1}} & P_0^{-1} \xrightarrow{d_0^{-1}} E^{-1} \longrightarrow 0 \\
& & \searrow h_{-3}^{-1} & \downarrow \beta_{-2}^{-1} & \searrow h_{-2}^{-1} & \downarrow \beta_{-1}^{-1} & \searrow h_{-1}^{-1} \downarrow \beta_0^{-1} \\
& & & & & & & \downarrow 0 \\
\cdots & \xrightarrow{\Phi(d_{-3}^{-1})} & \Phi(P_{-2}^{-1}) & \xrightarrow{\Phi(d_{-2}^{-1})} & \Phi(P_{-1}^{-1}) & \xrightarrow{\Phi(d_{-1}^{-1})} & \Phi(P_0^{-1}) \xrightarrow{\Phi(d_0^{-1})} \Phi(E^{-1}) \longrightarrow 0
\end{array}$$

where $\beta_j^{-1} = w_{P_j^{-1}} - \Phi(\phi_j^{-1}) \circ \phi_j^0$.

Assume that \mathcal{A} has small products and define the following two objects of \mathcal{A} by combining even and odd components of the two resolutions:

$$\begin{aligned}
P^{-1} &= \prod_{2l} \Phi^{-l}(P_{2l}^{-1}) \oplus \prod_{2l+1} \Phi^{-l-1}(P_{2l+1}^0) \\
P^0 &= \prod_{2l} \Phi^{-l}(P_{2l}^0) \oplus \prod_{2l+1} \Phi^{-l}(P_{2l+1}^{-1}).
\end{aligned}$$

We will denote the components of a factorization, $(P^{-1}, P^0, \phi_P^{-1}, \phi_P^0)$, we denote the components,

$$\begin{aligned}
\phi_{2l+1,2j+1}^{-1} &: \Phi^{-l-1}(P_{2l+1}^{-1}) \rightarrow \Phi^{-1}(P^0) \xrightarrow{\phi_P^{-1}} P^{-1} \rightarrow \Phi^{-j-1}(P_{2j+1}^0) \\
\phi_{2l+1,2j}^{-1} &: \Phi^{-l-1}(P_{2l+1}^{-1}) \rightarrow \Phi^{-1}(P^0) \xrightarrow{\phi_P^{-1}} P^{-1} \rightarrow \Phi^{-j}(P_{2j}^{-1}) \\
\phi_{2l,2j+1}^{-1} &: \Phi^{-l-1}(P_{2l}^0) \rightarrow \Phi^{-1}(P^0) \xrightarrow{\phi_P^{-1}} P^{-1} \rightarrow \Phi^{-j-1}(P_{2j+1}^0) \\
\phi_{2l,2j}^{-1} &: \Phi^{-l-1}(P_{2l}^0) \rightarrow \Phi^{-1}(P^0) \xrightarrow{\phi_P^{-1}} P^{-1} \rightarrow \Phi^{-j}(P_{2j}^{-1})
\end{aligned}$$

and

$$\begin{aligned}
\phi_{2l+1,2j+1}^0 &: \Phi^{-l-1}(P_{2l+1}^0) \rightarrow P^{-1} \xrightarrow{\phi_P^0} P^0 \rightarrow \Phi^{-j}(P_{2j+1}^{-1}) \\
\phi_{2l+1,2j}^0 &: \Phi^{-l-1}(P_{2l+1}^0) \rightarrow P^{-1} \xrightarrow{\phi_P^0} P^0 \rightarrow \Phi^{-j}(P_{2j}^0) \\
\phi_{2l,2j+1}^0 &: \Phi^{-l}(P_{2l}^{-1}) \rightarrow P^{-1} \xrightarrow{\phi_P^0} P^0 \rightarrow \Phi^{-j}(P_{2j+1}^{-1}) \\
\phi_{2l,2j}^0 &: \Phi^{-l}(P_{2l}^{-1}) \rightarrow P^{-1} \xrightarrow{\phi_P^0} P^0 \rightarrow \Phi^{-j}(P_{2j}^0).
\end{aligned}$$

Theorem 3.4. *Assume that small products exist in \mathcal{A} and that \mathcal{A} has enough projective objects.*

Let E be an object of $\text{Fact}(w)$. Choose projective resolutions of its components, lifts of ϕ_E^{-1} and ϕ_E^0 to these projective resolutions, and null-homotopies of the difference of w and the compositions of the lifts as above. Define P^{-1} and P^0 as above.

There exists a factorization, $P = (P^{-1}, P^0, \phi_P^{-1}, \phi_P^0)$, and a contra-quasi-isomorphism, $d_0 : P \rightarrow E$, such that

- *We have $\phi_{p,q}^{-1} = \phi_{p,q}^0 = 0$ for $q > p + 1$.*

- We have equalities,

$$\begin{aligned}\Phi^{-l-1}(d_{2l+2}^{-1}) &= \phi_{2l+1,2l+2}^{-1} : \Phi^{-l-1}(P_{2l+1}^{-1}) \rightarrow \Phi^{-l-1}(P_{2l+2}^{-1}) \\ -\Phi^{-l-1}(d_{2l+1}^0) &= \phi_{2l,2l+1}^{-1} : \Phi^{-l-1}(P_{2l}^0) \rightarrow \Phi^{-l-1}(P_{2l+1}^0) \\ -\Phi^{-l-1}(d_{2l+2}^0) &= \phi_{2l+1,2l+2}^0 : \Phi^{-l-1}(P_{2l+1}^0) \rightarrow \Phi^{-l-1}(P_{2l+2}^0) \\ \Phi^{-l}(d_{2l+1}^{-1}) &= \phi_{2l,2l+1}^0 : \Phi^{-l}(P_{2l}^{-1}) \rightarrow \Phi^{-l}(P_{2l+1}^{-1}).\end{aligned}$$

- We have equalities,

$$\begin{aligned}\Phi^{-l-1}(\phi_{2l+1}^0) &= \phi_{2l+1,2l+1}^{-1} : \Phi^{-l-1}(P_{2l+1}^{-1}) \rightarrow \Phi^{-l-1}(P_{2l+1}^0) \\ \Phi^{-l}(\phi_{2l}^{-1}) &= \phi_{2l,2l}^{-1} : \Phi^{-l-1}(P_{2l}^0) \rightarrow \Phi^{-l}(P_{2l}^{-1}) \\ \Phi^{-l}(\phi_{2l+1}^{-1}) &= \phi_{2l+1,2l+1}^0 : \Phi^{-l-1}(P_{2l+1}^0) \rightarrow \Phi^{-l}(P_{2l+1}^{-1}) \\ \Phi^{-l}(\phi_{2l}^0) &= \phi_{2l,2l}^0 : \Phi^{-l}(P_{2l}^{-1}) \rightarrow \Phi^{-l}(P_{2l}^0).\end{aligned}$$

- We have equalities,

$$\begin{aligned}\Phi^{-l-1}(h_{2l}^{-1}) &= \phi_{2l+1,2l}^{-1} : \Phi^{-l-1}(P_{2l+1}^{-1}) \rightarrow \Phi^{-l}(P_{2l}^{-1}) \\ -\Phi^{-l}(h_{2l-1}^0) &= \phi_{2l,2l-1}^{-1} : \Phi^{-l-1}(P_{2l}^0) \rightarrow \Phi^{-l}(P_{2l-1}^0) \\ -\Phi^{-l}(h_{2l}^0) &= \phi_{2l+1,2l}^0 : \Phi^{-l-1}(P_{2l+1}^0) \rightarrow \Phi^{-l}(P_{2l}^0) \\ \Phi^{-l}(h_{2l-1}^{-1}) &= \phi_{2l,2l-1}^0 : \Phi^{-l}(P_{2l}^{-1}) \rightarrow \Phi^{-l+1}(P_{2l-1}^{-1}).\end{aligned}$$

- d_0 is given by the compositions,

$$\begin{aligned}P^{-1} &\xrightarrow{d_0^{-1}} P_0^{-1} \rightarrow E^{-1} \\ P^0 &\xrightarrow{d_0^0} P_0^0 \rightarrow E^0.\end{aligned}$$

- d_0 is a quasi-isomorphism when both injective resolutions are finite.

Furthermore, if

$$\begin{aligned}h_p^{-1} \circ \phi_{p+1}^{-1} &= \Phi(\phi_p^{-1}) \circ h_p^0 \\ \Phi(h_p^0) \circ \phi_{p+1}^0 &= \Phi(\phi_p^0) \circ h_p^{-1} \\ \Phi(h_{p-1}^{-1}) \circ h_p^{-1} &= 0 \\ \Phi(h_{p-1}^0) \circ h_p^0 &= 0.\end{aligned}$$

Then, we have

$$\phi_{p,q}^{-1} = \phi_{p,q}^0 = 0$$

for $q < p - 1$.

Proof. The proof is the dual to those for Theorem 3.1 and Corollary 3.2. □

4. SOME APPLICATIONS OF THE RESOLUTIONS

Assume we have two triples, (\mathcal{A}, Φ, w) and (\mathcal{B}, Ψ, v) , and an additive functor, $\theta : \mathcal{A} \rightarrow \mathcal{B}$, such that

$$\theta \circ \Phi \cong \Psi \circ \theta$$

and

$$\theta(w_A) = v_{\theta(A)} : \theta(A) \rightarrow \theta(\Phi(A)) \cong \Psi(\theta(A)).$$

for all objects, $A \in \mathcal{A}$.

Assume that θ is left-exact, that \mathcal{A} has small coproducts and enough injectives, and that coproducts of injectives are injective. Let

$$\mathbf{R}\theta : D(\mathcal{A}) \rightarrow D(\mathcal{B})$$

be the right-derived functor of θ .

Definition 4.1. We also have a functor on factorization categories,

$$\begin{aligned} \theta : \text{Fact}(w) &\rightarrow \text{Fact}(v) \\ E &\mapsto \theta(E) := (\theta(E^{-1}), \theta(E^0), \theta(\phi_E^{-1}), \theta(\phi_E^0)). \end{aligned}$$

The right co-derived functor of θ on factorizations is defined to be the composition,

$$K(\text{Inj } w) \xrightarrow{\theta} K(\text{Fact } v) \rightarrow D^{\text{co}}(\text{Fact } v).$$

By Proposition 2.15, we get a functor,

$$\mathbf{R}\theta : D^{\text{co}}(\text{Fact } w) \rightarrow D^{\text{co}}(\text{Fact } v).$$

Lemma 4.2. *Assume that \mathcal{B} has finite injective dimension. Let E and F be objects of $\text{Fact}(w)$ whose components have finite injective dimension. If the map,*

$$\mathbf{R}\theta : \text{Hom}_{D(\mathcal{A})}(E^i, F^j[t]) \rightarrow \text{Hom}_{D(\mathcal{B})}(\mathbf{R}\theta(E^i), \mathbf{R}\theta(F^j)[t]),$$

is an isomorphism for all $i, j, t \in \mathbb{Z}$, then the map,

$$\mathbf{R}\theta : \text{Hom}_{D^{\text{co}}(\text{Fact } w)}(E, F[t]) \rightarrow \text{Hom}_{D^{\text{co}}(\text{Fact } v)}(\mathbf{R}\theta(E), \mathbf{R}\theta(F)[t]),$$

is an isomorphism for all $t \in \mathbb{Z}$

The proof of Lemma 4.2 will be a direct result of studying a spectral sequence associated to a filtration on morphism complexes, Hom^* . Before presenting it, let us first recall one method for computing the maps,

$$\mathbf{R}\theta : \text{Hom}_{D(\mathcal{A})}(A, A') \rightarrow \text{Hom}_{D(\mathcal{B})}(\mathbf{R}\theta(A'), \mathbf{R}\theta(A)),$$

on the ordinary derived categories.

Let C, D be chain complexes from \mathcal{A} . We have the chain complex,

$$\text{Hom}_{\mathcal{A}}^n(C, D) = \prod_{j-i=n} \text{Hom}_{\mathcal{A}}(C^i, D^j),$$

with

$$d(\prod_i g^i : C^i \rightarrow D^{i+n}) := \prod_i (d_D^{i+n+1} \circ g^i - (-1)^n g^{i+1} \circ d_C^{i+1}).$$

First, we choose injective resolutions,

$$\begin{aligned} 0 \rightarrow A' &\xrightarrow{d'_0} I'_0 \xrightarrow{d'_1} I'_1 \xrightarrow{d'_2} I'_2 \xrightarrow{d'_3} \dots \\ 0 \rightarrow A &\xrightarrow{d_0} I_0 \xrightarrow{d_1} I_1 \xrightarrow{d_2} I_2 \xrightarrow{d_3} \dots \end{aligned}$$

Next, we construct a commutative diagram,

$$\begin{array}{ccccccc}
\theta(I_0) & \xrightarrow{\theta(d_1)} & \theta(I_1) & \xrightarrow{\theta(d_2)} & \theta(I_2) & \xrightarrow{\theta(d_3)} & \theta(I_3) \xrightarrow{\theta(d_4)} \dots \\
\downarrow d_{0,0}^v & & \downarrow d_{1,0}^v & & \downarrow d_{2,0}^v & & \downarrow d_{3,0}^v \\
J_{0,0} & \xrightarrow{d_{1,0}^h} & J_{1,0} & \xrightarrow{d_{2,0}^h} & J_{2,0} & \xrightarrow{d_{3,0}^h} & J_{3,0} \xrightarrow{d_{4,0}^h} \dots \\
\downarrow d_{0,1}^v & & \downarrow d_{1,1}^v & & \downarrow d_{2,1}^v & & \downarrow d_{3,1}^v \\
J_{0,1} & \xrightarrow{d_{1,1}^h} & J_{1,1} & \xrightarrow{d_{2,1}^h} & J_{2,1} & \xrightarrow{d_{3,1}^h} & J_{3,1} \xrightarrow{d_{4,1}^h} \dots \\
\downarrow d_{0,2}^v & & \downarrow d_{1,2}^v & & \downarrow d_{2,2}^v & & \downarrow d_{3,2}^v \\
J_{0,2} & \xrightarrow{d_{1,2}^h} & J_{1,2} & \xrightarrow{d_{2,2}^h} & J_{2,2} & \xrightarrow{d_{3,2}^h} & J_{3,2} \xrightarrow{d_{4,2}^h} \dots \\
\downarrow d_{0,3}^v & & \downarrow d_{1,3}^v & & \downarrow d_{2,3}^v & & \downarrow d_{3,3}^v \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

where the rows and columns are exact chain complexes, all squares commute, and each $J_{p,q}$ is injective. Form the associated total complex, $J = (J_*, d_*^t)$, where

$$J_r = \bigoplus_{p+q=r} J_{p,q}$$

and the differential,

$$d_r^t : J_r \rightarrow J_{r+1},$$

is the product of the maps,

$$J_{p,q} \xrightarrow{d_{p+1,q}^h \oplus (-1)^q d_{p,q+1}^v} J_{p+1,q} \oplus J_{p,q+1}.$$

This comes with a map of chain complexes, $\theta(I) \rightarrow J$.

The map,

$$\mathbf{R}\theta : \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(A', A[t]) \rightarrow \mathrm{Hom}_{\mathbf{D}(\mathcal{B})}(\mathbf{R}\theta(A'), \mathbf{R}\theta(A)[t]),$$

is the cohomology in degree t of the map of chain complexes,

$$\mathrm{Hom}_{\mathcal{A}}^*(I', I) \xrightarrow{\theta} \mathrm{Hom}_{\mathcal{A}}^*(\theta(I'), \theta(I)) \rightarrow \mathrm{Hom}_{\mathcal{A}}^*(\theta(I'), J).$$

With this recap fresh in our mind, let us proceed with the proof of Lemma 4.2.

Proof of Lemma 4.2. Choose finite injective resolutions of the components,

$$\begin{aligned}
0 &\rightarrow E^{-1} \xrightarrow{d_0^{E^{-1}}} I_0^{E^{-1}} \xrightarrow{d_1^{E^{-1}}} I_1^{E^{-1}} \xrightarrow{d_2^{E^{-1}}} I_2^{E^{-1}} \xrightarrow{d_3^{E^{-1}}} \dots \\
0 &\rightarrow E^0 \xrightarrow{d_0^{E^0}} I_0^{E^0} \xrightarrow{d_1^{E^0}} I_1^{E^0} \xrightarrow{d_2^{E^0}} I_2^{E^0} \xrightarrow{d_3^{E^0}} \dots \\
0 &\rightarrow F^{-1} \xrightarrow{d_0^{F^{-1}}} I_0^{F^{-1}} \xrightarrow{d_1^{F^{-1}}} I_1^{F^{-1}} \xrightarrow{d_2^{F^{-1}}} I_2^{F^{-1}} \xrightarrow{d_3^{F^{-1}}} \dots \\
0 &\rightarrow F^0 \xrightarrow{d_0^{F^0}} I_0^{F^0} \xrightarrow{d_1^{F^0}} I_1^{F^0} \xrightarrow{d_2^{F^0}} I_2^{F^0} \xrightarrow{d_3^{F^0}} \dots,
\end{aligned}$$

and apply Theorem 3.1 to get resolutions of factorizations,

$$\begin{aligned} E &\rightarrow I^E \\ F &\rightarrow I^F. \end{aligned}$$

Recall that the components of I^F are

$$\begin{aligned} (I^F)^{-1} &= \bigoplus_{2l} \Phi^{-l}(I_{2l}^{-1}) \oplus \bigoplus_{2l+1} \Phi^{-l-1}(I_{2l+1}^0) \\ (I^F)^0 &= \bigoplus_{2l} \Phi^{-l}(I_{2l}^0) \oplus \bigoplus_{2l+1} \Phi^{-l}(I_{2l+1}^{-1}). \end{aligned}$$

Applying θ , we get the factorization, $\theta(I^F)$, whose components are

$$\begin{aligned} \theta(I^F)^{-1} &= \bigoplus_{2l} \theta(\Phi^{-l}(I_{2l}^{-1})) \oplus \bigoplus_{2l+1} \theta(\Phi^{-l-1}(I_{2l+1}^0)) \\ \theta(I^F)^0 &= \bigoplus_{2l} \theta(\Phi^{-l}(I_{2l}^0)) \oplus \bigoplus_{2l+1} \theta(\Phi^{-l}(I_{2l+1}^{-1})). \end{aligned}$$

We want to replace $\theta(I^F)$ by an injective factorization to compute $\mathbf{R}\theta$. We will apply Theorem 3.1, but, first, we need to choose injective resolutions of the components of $\theta(I^F)$.

To do this, we first construct finite diagrams (which exist by assumption),

$$\begin{array}{ccccccc} \theta(I_0^{F^{-1}}) & \xrightarrow{\theta(d_1^{F^{-1}})} & \theta(I_1^{F^{-1}}) & \xrightarrow{\theta(d_2^{F^{-1}})} & \theta(I_2^{F^{-1}}) & \xrightarrow{\theta(d_3^{F^{-1}})} & \dots \\ \downarrow d_{0,0}^{v,F^{-1}} & & \downarrow d_{1,0}^{v,F^{-1}} & & \downarrow d_{2,0}^{v,F^{-1}} & & \downarrow d_{3,0}^{v,F^{-1}} \\ J_{0,0}^{F^{-1}} & \xrightarrow{d_{1,0}^{h,F^{-1}}} & J_{1,0}^{F^{-1}} & \xrightarrow{d_{2,0}^{h,F^{-1}}} & J_{2,0}^{F^{-1}} & \xrightarrow{d_{3,0}^{h,F^{-1}}} & \dots \\ \downarrow d_{0,1}^{v,F^{-1}} & & \downarrow d_{1,1}^{v,F^{-1}} & & \downarrow d_{2,1}^{v,F^{-1}} & & \downarrow d_{3,1}^{v,F^{-1}} \\ J_{0,1}^{F^{-1}} & \xrightarrow{d_{1,1}^{h,F^{-1}}} & J_{1,1}^{F^{-1}} & \xrightarrow{d_{2,1}^{h,F^{-1}}} & J_{2,1}^{F^{-1}} & \xrightarrow{d_{3,1}^{h,F^{-1}}} & \dots \\ \downarrow d_{0,2}^{v,F^{-1}} & & \downarrow d_{1,2}^{v,F^{-1}} & & \downarrow d_{2,2}^{v,F^{-1}} & & \downarrow \vdots \\ \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

and

$$\begin{array}{ccccccc}
\theta(I_0^{F^{-1}}) & \xrightarrow{\theta(d_1^{F^0})} & \theta(I_1^{F^{-1}}) & \xrightarrow{\theta(d_2^{F^0})} & \theta(I_2^{F^{-1}}) & \xrightarrow{\theta(d_3^{F^0})} & \dots \\
d_{0,0}^{v,F^0} \downarrow & & d_{1,0}^{v,F^0} \downarrow & & d_{2,0}^{v,F^0} \downarrow & & \\
J_{0,0}^{F^{-1}} & \xrightarrow{d_{1,0}^{h,F^0}} & J_{1,0}^{F^{-1}} & \xrightarrow{d_{2,0}^{h,F^0}} & J_{2,0}^{F^{-1}} & \xrightarrow{d_{3,0}^{h,F^0}} & \dots \\
d_{0,1}^{v,F^0} \downarrow & & d_{1,1}^{v,F^0} \downarrow & & d_{2,1}^{v,F^0} \downarrow & & \\
J_{0,1}^{F^{-1}} & \xrightarrow{d_{1,1}^{h,F^0}} & J_{1,1}^{F^{-1}} & \xrightarrow{d_{2,1}^{h,F^0}} & J_{2,1}^{F^{-1}} & \xrightarrow{d_{3,1}^{h,F^0}} & \dots \\
d_{0,2}^{v,F^0} \downarrow & & d_{1,2}^{v,F^0} \downarrow & & d_{2,2}^{v,F^0} \downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

where the rows and columns are exact, all J 's are injective, and all squares commute. Then, we use the injective resolutions,

$$\begin{aligned}
0 \rightarrow \theta(I^F)^{-1} \oplus_{2l} \Psi^{-l}(d_{0,2l}^{v,F^{-1}}) \oplus_{\rightarrow} \oplus_{2l+1} \Psi^{-l-1}(d_{0,2l+1}^{v,F^0}) \bigoplus_{2l} \Psi^{-l}(J_{0,2l}^{F^{-1}}) \oplus \bigoplus_{2l+1} \Psi^{-l-1}(J_{0,2l+1}^{F^0}) \\
\oplus_{2l} \Psi^{-l}(d_{1,2l}^{v,F^{-1}}) \oplus_{\rightarrow} \oplus_{2l+1} \Psi^{-l-1}(d_{1,2l+1}^{v,F^0}) \bigoplus_{2l} \Psi^{-l}(J_{1,2l}^{F^{-1}}) \oplus \bigoplus_{2l+1} \Psi^{-l-1}(J_{1,2l+1}^{F^0}) \rightarrow \dots
\end{aligned}$$

and

$$\begin{aligned}
0 \rightarrow \theta(I^F)^0 \oplus_{2l} \Psi^{-l}(d_{0,2l}^{v,F^0}) \oplus_{\rightarrow} \oplus_{2l+1} \Psi^{-l}(d_{0,2l+1}^{v,F^{-1}}) \bigoplus_{2l} \Psi^{-l}(J_{0,2l}^{F^0}) \oplus \bigoplus_{2l+1} \Psi^{-l}(J_{0,2l+1}^{F^{-1}}) \\
\oplus_{2l} \Psi^{-l}(d_{1,2l}^{v,F^0}) \oplus_{\rightarrow} \oplus_{2l+1} \Psi^{-l}(d_{1,2l+1}^{v,F^{-1}}) \bigoplus_{2l} \Psi^{-l}(J_{1,2l}^{F^0}) \oplus \bigoplus_{2l+1} \Psi^{-l}(J_{1,2l+1}^{F^{-1}}) \rightarrow \dots
\end{aligned}$$

and apply Theorem 3.1. Denote the resulting factorization by J . Note that the components of J are

$$\begin{aligned}
J^{-1} &= \bigoplus_{r+s=2l} \Psi^{-l}(J_{r,s}^{F^{-1}}) \oplus \bigoplus_{r+s=2l+1} \Psi^{-l-1}(J_{r,s}^{F^0}) \\
J^0 &= \bigoplus_{r+s=2l} \Psi^{-l}(J_{r,s}^{F^0}) \oplus \bigoplus_{r+s=2l+1} \Psi^{-l}(J_{r,s}^{F^{-1}}).
\end{aligned}$$

The chain complex, $\mathrm{Hom}_v^*(\theta(I^E), J)$, admits a filtration,

$$\begin{aligned} \mathcal{F}^p \mathrm{Hom}_v^n(\theta(I^E), J) &:= \{(g^{-1}, g^0) \mid \forall t \\ &\left\{ \begin{array}{ll} g^{-1} \left(\bigoplus_{2l \leq t-1} \Psi^{-l}(\theta(I_{2l}^{E^{-1}})) \oplus \bigoplus_{2l+1 \leq t-1} \Psi^{-l-1}(\theta(I_{2l+1}^{E^0})) \right) \\ \subseteq \bigoplus_{r+s=2l \leq t+p+n-1} \Psi^{m-l}(J_{r,s}^{F^{-1}}) \oplus \bigoplus_{r+s=2l+1 \leq t+p+n-1} \Psi^{m-l-1}(J_{r,s}^{F^0}) \\ g^0 \left(\bigoplus_{2l \leq t} \Psi^{-l}(\theta(I_{2l}^{E^0})) \oplus \bigoplus_{2l+1 \leq t} \Psi^{-l}(\theta(I_{2l+1}^{E^{-1}})) \right) \\ \subseteq \bigoplus_{r+s=2l \leq t+p+n} \Psi^{m-l}(J_{r,s}^{F^0}) \oplus \bigoplus_{r+s=2l+1 \leq t+p+n} \Psi^{m-l}(J_{r,s}^{F^{-1}}) \} & n = 2m \\ g^{-1} \left(\bigoplus_{2l \leq t-1} \Psi^{-l}(\theta(I_{2l}^{E^{-1}})) \oplus \bigoplus_{2l+1 \leq t-1} \Psi^{-l-1}(\theta(I_{2l+1}^{E^0})) \right) \\ \subseteq \bigoplus_{r+s=2l \leq t+p+n-1} \Psi^{m-l}(J_{r,s}^{F^0}) \oplus \bigoplus_{r+s=2l+1 \leq t+p+n-1} \Psi^{m-l}(J_{r,s}^{F^{-1}}) \\ g^0 \left(\bigoplus_{2l \leq t} \Psi^{-l}(\theta(I_{2l}^{E^0})) \oplus \bigoplus_{2l+1 \leq t} \Psi^{-l}(\theta(I_{2l+1}^{E^{-1}})) \right) \\ \subseteq \bigoplus_{r+s=2l \leq t+p+n} \Psi^{m-l+1}(J_{r,s}^{F^{-1}}) \oplus \bigoplus_{r+s=2l+1 \leq t+p+n} \Psi^{m-l}(J_{r,s}^{F^0}) \} & n = 2m + 1 \end{array} \right. \end{aligned}$$

After recombining even and odd parts, the associated graded complex is

$$\begin{aligned} \mathrm{Gr}^p \mathrm{Hom}_v^n(\theta(I^E), J) &= \\ &\left\{ \begin{array}{ll} \mathrm{Hom}_{\mathcal{B}}^{p+n}(\theta(I^{E^0}), \Psi^{-q}(J^{F^0})) \oplus \mathrm{Hom}_{\mathcal{B}}^{p+n}(\theta(I^{E^{-1}}), \Psi^{-q}(J^{F^{-1}})) & p = 2q \\ \mathrm{Hom}_{\mathcal{B}}^{p+n}(\theta(I^{E^0}), \Psi^{-q}(J^{F^{-1}})) \oplus \mathrm{Hom}_{\mathcal{B}}^{p+n}(\theta(I^{E^{-1}}), \Psi^{-q-1}(J^{F^0})) & p = 2q + 1 \end{array} \right. \end{aligned}$$

with the differential being the sum of the differentials on $\mathrm{Hom}_{\mathcal{B}}^*(\theta(I^{E^u}), \Psi^{-q}(J^{F^v}))$, $u, v \in \{-1, 0\}$.

There exists an analogous filtration on $\mathrm{Hom}_v^*(I^E, I^F)$ whose associated graded complex is

$$\left\{ \begin{array}{ll} \mathrm{Hom}_{\mathcal{A}}^{p+n}(I^{E^0}, \Phi^{-q}(I^{F^0})) \oplus \mathrm{Hom}_{\mathcal{A}}^{p+n}(I^{E^{-1}}, \Phi^{-q}(I^{F^{-1}})) & p = 2q \\ \mathrm{Hom}_{\mathcal{A}}^{p+n}(I^{E^0}, \Phi^{-q}(I^{F^{-1}})) \oplus \mathrm{Hom}_{\mathcal{A}}^{p+n}(I^{E^{-1}}, \Phi^{-q-1}(I^{F^0})) & p = 2q + 1 \end{array} \right.$$

These filtrations are compatible with the map,

$$\mathrm{Hom}_v^*(I^E, I^F) \rightarrow \mathrm{Hom}_v^*(\theta(I^E), J).$$

The map on the associated graded complexes,

$$\mathrm{Gr}^p \mathrm{Hom}_v^*(I^E, I^F) \rightarrow \mathrm{Gr}^p \mathrm{Hom}_w^*(\theta(I^E), J^F),$$

is exactly the sum of the maps,

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}^*(I^{E^0}, \Phi^{-q}(I^{F^0})) &\rightarrow \mathrm{Hom}_{\mathcal{B}}^*(\theta(I^{E^0}), \Psi^{-q}(J^{F^0})) \\ \mathrm{Hom}_{\mathcal{A}}^*(I^{E^0}, \Phi^{-q}(I^{F^{-1}})) &\rightarrow \mathrm{Hom}_{\mathcal{B}}^*(\theta(I^{E^0}), \Psi^{-q}(J^{F^{-1}})) \\ \mathrm{Hom}_{\mathcal{A}}^*(I^{E^{-1}}, \Phi^{-q-1}(I^{F^0})) &\rightarrow \mathrm{Hom}_{\mathcal{B}}^*(\theta(I^{E^{-1}}), \Psi^{-q-1}(J^{F^0})) \\ \mathrm{Hom}_{\mathcal{A}}^*(I^{E^{-1}}, \Phi^{-q}(I^{F^{-1}})) &\rightarrow \mathrm{Hom}_{\mathcal{B}}^*(\theta(I^{E^{-1}}), \Psi^{-q}(J^{F^{-1}})), \end{aligned}$$

which we have assumed to be quasi-isomorphisms. The corresponding map of spectral sequences is an isomorphism on the E_1 -page. Since the injective resolutions are assumed to be finite, the spectral sequence degenerates and yields the desired statement. \square

Next assume that θ is right-exact, commutes with products, and \mathcal{A} has finite projective dimension. We have the usual left-derived functor,

$$\mathbf{L}\theta : \mathrm{D}(\mathcal{A}) \rightarrow \mathrm{D}(\mathcal{B}).$$

Definition 4.3. And, we have a left-contra-derived functor given by the composition,

$$K(\text{Proj } w) \xrightarrow{\theta} K(\text{Fact } v) \rightarrow D^{\text{ctr}}(\text{Fact } v).$$

Using Proposition 2.15, we get a functor,

$$\mathbf{L}\theta : D^{\text{ctr}}(\text{Fact } W) \rightarrow D^{\text{ctr}}(\text{Fact } v).$$

Lemma 4.4. *Let E and F be objects of $\text{Fact}(w)$. The map,*

$$\mathbf{L}\theta : \text{Hom}_{D^{\text{ctr}}(\text{Fact } w)}(E, F[t]) \rightarrow \text{Hom}_{D^{\text{ctr}}(\text{Fact } v)}(\mathbf{L}\theta E, \mathbf{L}\theta F[t]),$$

is an isomorphism for all $t \in \mathbb{Z}$ if the map

$$\mathbf{L}\theta : \text{Hom}_{D(\mathcal{A})}(E^i, F^j[t]) \rightarrow \text{Hom}_{D(\mathcal{B})}(\mathbf{L}\theta E^i, \mathbf{L}\theta F^j[t]),$$

is an isomorphism for all $i, j, t \in \mathbb{Z}$.

Proof. The proof of this lemma is completely analogous to the proof of Lemma 4.2. \square

We may also give a useful spectral sequence for computing morphisms in $D^{\text{abs}}(\text{Fact}(w))$.

Lemma 4.5. *Let E and F be two factorizations of w . Assume that \mathcal{A} has enough injectives and small coproducts, and assume that coproducts of injectives are injective.*

There is a spectral sequence whose E_1 -page is

$$E_1^{p,q} = \begin{cases} \text{Ext}_{\mathcal{A}}^{p+q-1}(E^{-1}, \Phi^{-s}(F^0)) \oplus \text{Ext}_{\mathcal{A}}^{p+q}(E^0, \Phi^{-s}(F^0)) & p = 2s \\ \text{Ext}_{\mathcal{A}}^{p+q-1}(E^{-1}, \Phi^{-s}(F^{-1})) \oplus \text{Ext}_{\mathcal{A}}^{p+q}(E^0, \Phi^{-s-1}(F^{-1})) & p = 2s + 1. \end{cases}$$

If the components of F have finite injective dimension, the spectral sequence strongly converges to $\bigoplus_r \text{Hom}_{D^{\text{co}}(\text{Fact } w)}(E, F[r])$.

Proof. Choose finite injective resolutions of F^{-1} and F^0 ,

$$\begin{aligned} 0 \rightarrow F^{-1} \rightarrow I_0^{-1} \rightarrow I_1^{-1} \rightarrow \dots \\ 0 \rightarrow F^0 \rightarrow I_0^0 \rightarrow I_1^0 \rightarrow \dots, \end{aligned}$$

and use Theorem 3.1 to construct a co-quasi-isomorphic resolution, I , of F .

Filter the complex, $\text{Hom}_w^*(E, I)$, by

$$\begin{aligned} \mathcal{F}^p \text{Hom}_w^n(E, I) &:= \{(g^{-1}, g^0) \mid \\ &\begin{cases} g^{-1}(E^{-1}) \subseteq \bigoplus_{2l \leq n+p-1} \Phi^{m-l}(I_{2l}^{-1}) \oplus \bigoplus_{2l+1 \leq n+p-1} \Phi^{m-l-1}(I_{2l+1}^0) \\ g^0(E^0) \subseteq \bigoplus_{2l \leq n+p} \Phi^{m-l}(I_{2l}^0) \oplus \bigoplus_{2l+1 \leq n+p} \Phi^{m-l}(I_{2l+1}^{-1}) \\ g^{-1}(E^{-1}) \subseteq \bigoplus_{2l \leq n+p-1} \Phi^{m-l}(I_{2l}^0) \oplus \bigoplus_{2l+1 \leq n+p-1} \Phi^{m-l}(I_{2l+1}^{-1}) \\ g^0(E^0) \subseteq \bigoplus_{2l \leq n+p} \Phi^{m-l+1}(I_{2l}^{-1}) \oplus \bigoplus_{2l+1 \leq n+p} \Phi^{m-l}(I_{2l+1}^{-1}) \end{cases} \\ &\quad n = 2m \\ &\quad n = 2m + 1 \}. \end{aligned}$$

The associated graded complex is

$$\text{Gr}^p \text{Hom}_w^n(E, I) := \begin{cases} \text{Hom}_{\mathcal{A}}(E^{-1}, \Phi^{-q}(I_{p+n-1}^0)) \oplus \text{Hom}_{\mathcal{A}}(E^0, \Phi^{-q}(I_{p+n}^0)) & p = 2q \\ \text{Hom}_{\mathcal{A}}(E^{-1}, \Phi^{-q}(I_{p+n-1}^{-1})) \oplus \text{Hom}_{\mathcal{A}}(E^0, \Phi^{-q-1}(I_{p+n}^{-1})) & p = 2q + 1. \end{cases}$$

with differentials given by composition with the differentials in the complexes I_*^{-1} and I_*^0 .

We set

$$E_0^{p,q} := \text{Gr}^p \text{Hom}_w^q(E, I)$$

to start our spectral sequence. The E_1 -page is as above.

If we assume that the components of F have injective resolutions of length t , then the spectral sequence degenerates at the $(t + 1)$ -st page. \square

Lemma 4.6. *Let E and F be two factorizations of w . Assume that \mathcal{A} has enough projectives and small coproducts.*

There is a spectral sequence whose E_1 -page is

$$E_1^{p,q} = \begin{cases} \mathrm{Ext}_{\mathcal{A}}^{p+q-1}(E^{-1}, \Phi^{-s}(F^0)) \oplus \mathrm{Ext}_{\mathcal{A}}^{p+q}(E^0, \Phi^{-s}(F^0)) & p = 2s \\ \mathrm{Ext}_{\mathcal{A}}^{p+q-1}(E^{-1}, \Phi^{-s}(F^{-1})) \oplus \mathrm{Ext}_{\mathcal{A}}^{p+q}(E^0, \Phi^{-s-1}(F^{-1})) & p = 2s + 1. \end{cases}$$

If the components of E have finite projective dimension, the spectral sequence strongly converges to $\bigoplus_r \mathrm{Hom}_{\mathrm{D}^{\mathrm{ctr}}(\mathrm{Fact} w)}(E, F[r])$.

Proof. The proof is completely analogous to that of Lemma 4.5 \square

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